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Neutrino propagation in matter using the wave packet approach

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Abstract: We study the oscillations and conversions of relativistic neutrinos propagating in matter of variable density using the wave packet formalism. We show how the oscillation and coherence lengths are modified in comparison with the case of oscillations in vacuum. Secondly, we demonstrate how the equation of motion for two neutrino flavors can be formally solved for almost arbitrary density profile. We calculate finally how the use of wave packets alters the non-adiabatic level crossing probabilities. For the most common physical environments the corrections due to the width of the wave packet do not lead to observable effects.

Keywords: Solar and Atmospheric Neutrinos, Neutrino physics.

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Contents

1. Introduction

Recent experimental discoveries [1] seem to suggest that neutrino oscillations [2]–[5] really exist. Since the observation of neutrino oscillation may provide valuable information on the basic properties of neutrinos, e.g. masses and mixing angles, it is important to know the underlying physics also on the conceptual level.

It was pointed out e.g. in refs. [6] and [7] that the standard quantum mechanical treatment of neutrino oscillations using plane waves [5, 8] is not completely satisfactory for many reasons. The wave packet approach [6] and [9]–[15] provides a more physical picture which is particularly adapted for describing phenomena localized in space and time. This formalism also elegantly accounts for the loss of coherence by the separation of wave packets. However, some authors (e.g. [7, 16, 17]) have been skeptical about the use of wave packets, and refs. [17, 18] conclude that the concept of wave packet is unnecessary for all the relevant physical cases. A bunch of other methods has also been discussed [7] and [16]–[20] (to name a few).

In this paper we will consider neutrino oscillations and other phenomena in matter. We have decided to use wave packets because the calculations of the kind presented here have never been carried out before.

We will first focus on the equation of motion for neutrinos propagating in matter of variable density. Generally the equation is modified in matter due to coherent forward scatterings [21]–[24], and an exact solution can be found only for few special cases. Here we address one special case where the density of matter changes slowly enough, so that the situation is said to be adiabatic. Now the eigensolutions for the equation of motion are found trivially, for arbitrary number of relativistic neutrinos. To describe neutrino oscillations in matter, we apply the method of ref. [14] in connection with these solutions. We can express the results formally using effective oscillation and coherence lengths which are not local quantities anymore.

Generally neutrinos may propagate nonadiabatically, and solving the complete equation of motion even for two neutrino flavors is usually far from trivial (see e.g. [25, 26, 27] for two specific density profiles). In this paper we show that the solution for "arbitrary" density profile can be constructed by using infinite integral series. Some supplementary calculations which may be of formal interest are enclosed in appendix B. Nonadiabaticity is also related to the fact that so-called level crossings (or hoppings) between matter "eigenstates" take place ([24, 28] and references therein). Our calculations show that the level crossing probabilities are modified when neutrinos are described by wave packets.

2. Neutrino propagation in matter in the adiabatic limit

2.1 Equation of motion

Barring the details of the production and detection mechanisms of neutrinos, we will focus on the propagation. Traditionally the propagation of neutrinos is modeled by solving the relativistic Schrödinger equation with the effective hamiltonian

$$\hat{H} = \sqrt{\hat{p}^2 + m^2} + V(\hat{x}, t) \approx \hat{p} + \frac{1}{2} m^2 \hat{p}^{-1} + V(\hat{x}, t)\,, \tag{2.1}$$

where $\hat{p}$ is the momentum operator, m is the neutrino mass matrix and $V(x, t)$ is a semiclassical potential due to the presence of medium. In this work we assume that $V(x)$ does not depend on time, and that it is a matrix diagonal in the weak interaction basis or flavor basis. This is well justified when considering neutrinos from the sun, from supernovae, or on earth. On the other hand, this assumption excludes neutrinos in the early universe that must be treated otherwise.

Instead of decomposing the wave function in momentum space, we prefer to consider the Fourier transform in energy space,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int dE\, e^{-iEt} \psi(x, E)\,. \tag{2.2}$$

This leads to a more physical picture when the Hamilton operator is a function of x. Particularly, now the energy E is a constant, while the momentum p is a function of x. Throughout this work we will assume the propagation to be a one-dimensional phenomenon.

The resulting time independent Schrödinger equation can be written in the relativistic limit as

$$i\,\partial_x\,\psi(x,E) = \left(-E + \frac{m^2}{2E} + V(x)\right)\psi(x,E) + O\left(\frac{m^4 + m^2 E\,V}{E^3}\right)\psi(x,E)\,, \quad (2.3)$$

where we have also assumed that $E \gg V$. The right-hand side can be locally diagonalized. We call the respective eigenvectors as local matter eigenstates, and the respective eigenvalues can be written as

$$p_a(x,E) \equiv E - \frac{\mu_a^2(x,E)}{2E}\,, \quad (2.4)$$

where we introduced the effective mass $\mu_a(x,E)$ that simplifies the notation. In this paper the Greek indices refer to the flavor basis and the Latin indices to the matter basis. In the adiabatic limit, or for negligible mixing, eq. (2.3) can be written as

$$i\,\partial_x\,\psi_a(x,E) \approx \left(-E + \frac{\mu_a^2(x,E)}{2E}\right)\psi_a(x,E) = -p_a(x,E)\,\psi_a(x,E)\,. \quad (2.5)$$

Outside the adiabatic limit the complete equation of motion includes also non-diagonal terms (compare to eq. (11) of ref. [29]).

From now on we follow the treatment presented in ref. [14]. Equation (2.5) can be solved easily, and one has

$$\psi_a(x,t) = \frac{1}{\sqrt{2\pi}}\int dE\,\exp\left[i\int_0^x dx' p_a(x',E) - i\,E\,t\right]\psi_a(0,E)\,, \quad (2.6)$$

where eq. (2.2) was used. The initial value of x is put to zero since we consider neutrinos produced at the origin.

2.2 Wave packet solutions

As initial condition, we assume that the neutrino wave function is a linear combination of matter eigenstates,

$$\psi(0,E) = \sum_a C_a(E)\,\psi_a(0,E)\,. \quad (2.7)$$

In the relativistic limit the coefficients C_a match the respective row in the mixing matrix, $C_a(x,E) = U^*_{\alpha a}(x,E)$, but for less relativistic neutrinos correction factors related to energy conservation must be taken into account [30]. We assume further that each component of the wave function $\psi_a(0,E)$ has a gaussian form

$$\psi_a(0,E) = \left(2\pi\sigma_{EP}^2\right)^{-1/4}\exp\left[-\frac{(E-E_a)^2}{4\sigma_{EP}^2}\right]\,, \quad (2.8)$$

where E_a is the average energy of the corresponding matter eigenstate and σ_{EP} is the energy width related to the production process fulfilling the uncertainty relation $\sigma_{EP}\,\sigma_{tP} = 1/2$.

In the adiabatic limit we obtain for the partial wave packet the solution

$$\psi_a(x,t) = C_1 \int dE \exp\left[i\int_0^x dx' p_a(x',E) - i\,E\,t - \frac{(E-E_a)^2}{4\sigma_{EP}^2}\right], \tag{2.9}$$

where C_1 is a numerical factor. We assume that the wave packet is sufficiently narrow in energy space, $\sigma_{EP} \ll E_a$. This helps us to integrate the above integral and we also avoid considering the negative energies. This assumption is justified e.g. for solar neutrinos [12, 24]. One can now expand the momentum as

$$p_a(x,E) \approx p_a(x,E_a) + \frac{\partial p_a}{\partial E_a}(E-E_a) = p_a(x,E_a) + \frac{E-E_a}{v_a(x,E_a)}, \tag{2.10}$$

where $v_a(x,E)$ is the group velocity of each wave packet. Note that within this framework the group velocity depends on x via the potential $V(x)$, unlike in some alternative works. Writing $\sigma_{EP} \approx v_a(0)\,\sigma_{pP} = v_a(0)/(2\sigma_{xP})$, one has after a simple integration

$$\psi_a(x,t) = C_2 \exp\left[i\int_0^x dx' p_a(x',E_a) - i\,E_a\,t - \frac{v_a^2(0)}{4\sigma_{xP}^2}\left(\int_0^x \frac{dx'}{v_a(x')} - t\right)^2\right], \tag{2.11}$$

where $v_a(0)$ is the group velocity at the origin and $\sigma_{pP}(\sigma_{xP})$ is the momentum (spatial) width related to the production process. Equation (2.11) implies that the wave function of a flavor neutrino ν_α is given by [14]

$$\begin{aligned} |\nu_\alpha(x,t)\rangle = C_2 \sum_a U_{\alpha a}^*(0) \exp\Bigg[i\int_0^x dx' p_a(x',E_a) - i\,E_a\,t - \\ - \frac{v_a^2(0)}{4\sigma_{xP}^2}\left(\int_0^x \frac{dx'}{v_a(x')} - t\right)^2\Bigg] |\nu_a\rangle, \end{aligned} \tag{2.12}$$

where $U(x)$ represents the effective mixing matrix and the states $|\nu_a\rangle$ are orthonormal.

To get out relevant physics we need to know the size of the wave packet. This is very non trivial and lots of rather confusing estimates have been presented. Here we assume that the width of the neutrino wave is mostly related to the spatial details of the production process. In principle it also depends on temporal properties, like the stability of the state producing the neutrino, but this is relevant only for very short-lived particles. It has been estimated ([12, 24] and relevant references therein) that σ_{xP} is of the order 10^{-9} m for solar neutrinos, $\sigma_{xP} \sim 10^{-11}$ m at the neutrino sphere of a supernova, and $\sigma_{xP} \sim 10^{-6}$ m for reactor neutrinos, for example.

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2.3 Observation of neutrino oscillation

The quantum state of the neutrino is measured by an appropriate reaction. Assuming the respective process being $\nu_\alpha + X \to$ (something visible), we can relate the quantum mechanical uncertainty of the detection process to the quantum mechanical state of the particle X before the collision. Here we assume that the relevant wave functions are gaussians, centered at a distance L from the origin (source), with a spatial width σ_{xD}. Hence the detection can be described by

$$|\nu_\beta(x-L)\rangle = C_2' \sum_a U^*_{\beta a}(L) \exp\left[i \int_L^x dx'\, p_a(x', E_a) - \right.$$
$$\left. - \frac{v_a^2(L)}{4\sigma_{xD}^2} \left(\int_L^x \frac{dx'}{v_a(x')} \right)^2 \right] |\nu_a\rangle \,, \qquad (2.13)$$

where $v_a(L)$ is the group velocity at L. Note that this is independent of time [14].

We emphasize that the spatial uncertainty, σ_{xD}, arises for purely quantum-mechanical reasons. One may be apt to believe that it is at most of the order of atomic distances, i.e. $\sigma_{xD} \sim 10^{-9}\,\mathrm{m} - 10^{-10}\,\mathrm{m}$ or less. In reality there are also other instrumental uncertainties related to the resolution or dimension of the detector. These can be accounted for by taking an average over the relevant length scale.

The amplitude of the process $\nu_\alpha \to \nu_\beta$ is now simply

$$\begin{aligned} A_{\alpha\beta}(L,T) &= \int dx \Big\langle \nu_\beta(x-L) \mid \nu_\alpha(x,T) \Big\rangle \\ &= C_2'' \sum_a U^*_{\alpha a}(0)\, U_{\beta a}(L) \times \\ &\quad \times \int dx \exp\left[i \int_0^L dx' p_a(x', E_a) - i\, E_a\, T - \right. \qquad (2.14) \\ &\quad \left. - \frac{v_a^2(0)}{4\sigma_{xP}^2} \left(\int_0^x \frac{dx'}{v_a(x')} - T \right)^2 - \frac{v_a^2(L)}{4\sigma_{xD}^2} \left(\int_L^x \frac{dx'}{v_a(x')} \right)^2 \right]. \end{aligned}$$

The first two terms in the exponential are independent of x, but the integration of the two other terms is far from obvious. Notice that the analytical form of $v_a(x)$ is unknown in general, and that we have not even chosen any specific density profile so far. It turns out, however, that the integral can be evaluated with a saddle point method to a sufficiently good approximation.

After a straightforward, but rather lengthy calculation, presented in appendix A, we obtain the amplitude (with eqs. (A.6) and (A.10))

$$A_{\alpha\beta}(L,T) = C_3 \sum_a U^*_{\alpha a}(0)\, U_{\beta a}(L) \exp\left[\, i \int_0^L dx\, p_a(x, E_a) - i\, E_a\, T - \right.$$
$$\left. - \frac{1}{4} \frac{\left(T - \int_0^L \frac{dx}{v_a(x)} \right)^2}{\left(\frac{\sigma_{xP}}{v_a(0)} \right)^2 + \left(\frac{\sigma_{xD}}{v_a(L)} \right)^2} \right]. \qquad (2.15)$$

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The respective probability of the process $\nu_\alpha \to \nu_\beta$ is then

$$\begin{aligned} P_{\alpha\beta}(L,T) &= |A_{\alpha\beta}(L,T)|^2 \\ &\sim \sum_{a,b} U^*_{\alpha a}(0)\, U_{\beta a}(L)\, U_{\alpha b}(0)\, U^*_{\beta b}(L)\, e^{G(L,T)}\,, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} G(L,T) = i\int_0^L dx\Big(p_a(x,E_a) - p_b(x,E_b)\Big) - i\,(E_a - E_b)\,T - \\ - \frac{1}{4\chi_a}\left(T - \int_0^L \frac{dx}{v_a(x)}\right)^2 - \frac{1}{4\chi_b}\left(T - \int_0^L \frac{dx}{v_b(x)}\right)^2 \end{aligned} \tag{2.17}$$

with

$$\chi_a \equiv \left(\frac{\sigma_{xP}}{v_a(0)}\right)^2 + \left(\frac{\sigma_{xD}}{v_a(L)}\right)^2. \tag{2.18}$$

We still have to perform an integration over time in $P_{\alpha\beta}(L,T)$ since we are mainly interested in $P_{\alpha\beta}(L)$ [6, 14]. A straightforward gaussian integration of the exponential of eq. (2.16) yields

$$\begin{aligned} \int dT e^{G(L,T)} \sim \exp\left\{i\int_0^L dx\Big(p_a(x,E_a) - p_b(x,E_b)\Big)\right\} \times \\ \times \exp\left\{-\frac{1}{4\chi_a}\left(\int_0^L \frac{dx}{v_a(x)}\right)^2 - \frac{1}{4\chi_b}\left(\int_0^L \frac{dx}{v_b(x)}\right)^2\right\} \times \\ \times \exp\left\{\frac{\chi_a\,\chi_b}{\chi_a + \chi_b}\left[-(E_a - E_b)^2 + \right.\right. \\ + \frac{1}{4}\left(\frac{1}{\chi_a}\int_0^L \frac{dx}{v_a(x)} + \frac{1}{\chi_b}\int_0^L \frac{dx}{v_b(x)}\right)^2 - \\ \left.\left. - i\,(E_a - E_b)\left(\frac{1}{\chi_a}\int_0^L \frac{dx}{v_a(x)} + \frac{1}{\chi_b}\int_0^L \frac{dx}{v_b(x)}\right)\right]\right\} \end{aligned} \tag{2.19}$$

without the prefactor, which is a constant in the relativistic limit. A detailed analysis renders eq. (2.19) to

$$\begin{aligned} \int dT\, e^{G(L,T)} \sim \exp\left\{i\int_0^L dx\Big(p_a(x,E_a) - p_b(x,E_b)\Big) - i\,(E_a - E_b)\,L - \right. \\ \left. - \frac{1}{8\sigma_x^2}\left[\int_0^L dx(v_a(x) - v_b(x))\right]^2 - \frac{(E_a - E_b)^2}{8\,\sigma_p^2}\right\}, \end{aligned} \tag{2.20}$$

where $\sigma_x^2 \equiv \sigma_{xP}^2 + \sigma_{xD}^2, \sigma_p \equiv 1/(2\sigma_x)$ and the relativistic limit is once again considered.

In order to simplify eq. (2.20), we approximate (compare to ref. [14])

$$E_a \approx E_0 + \xi\,\frac{\mu_a^2(0)}{2E_0}\,, \tag{2.21}$$

where $\mu_a(0)$ is the effective mass at the origin, E_0 is the central energy in the limit of zero neutrino masses and ξ is a parameter of order unity, related to the energy-momentum conservation of the production process. The corresponding momentum is given by

$$p_a(x,E_a) \approx E_a - \frac{\mu_a^2(x)}{2E_a} \approx E_0 + \xi\,\frac{\mu_a^2(0)}{2E_0} - \frac{\mu_a^2(x)}{2E_0}\,. \tag{2.22}$$

Notice that E_a is a constant, but $p_a(x,E_a)$ depends on x, thus this approximation does not contradict the chosen convention. One may also point out that the initial dependence $\mu_a = \mu_a(x,E_a)$ has reduced to $\mu_a = \mu_a(x,E_0)$. Using eqs. (2.21) and (2.22), eq. (2.20) gives

$$\begin{aligned}\int dT e^{G(L,T)} \sim \exp\Bigg[&-\frac{i}{2E_0}\int_0^L dx\,\Delta\mu_{ab}^2(x) - \frac{1}{8\sigma_x^2}\left(\int_0^L dx \Delta v_{ab}(x)\right)^2 - \\ &- \frac{(E_a-E_b)^2}{8\,\sigma_p^2}\Bigg]\,,\end{aligned} \tag{2.23}$$

where $\Delta\mu_{ab}^2(x) \equiv \mu_a^2(x) - \mu_b^2(x)$, $\Delta v_{ab}(x) \equiv v_a(x) - v_b(x)$ and the last term has not been altered on purpose.

Combining the above results we obtain a compact formula for the probability to observe a neutrino in a state ν_β at a distance L (with the correct normalization $\sum_\beta P_{\alpha\beta}(L) = 1$),

$$\begin{aligned}P_{\alpha\beta}(L) = \sum_{a,b} & U_{\alpha a}^*(0)\,U_{\beta a}(L)\,U_{\alpha b}(0)\,U_{\beta b}^*(L)\times \\ &\times \exp\left[-2\pi i\,\frac{L}{L_{ab}^{\rm osc}(L)} - \left(\frac{L}{L_{ab}^{\rm coh}(L)}\right)^2 - \frac{(E_a-E_b)^2}{8\,\sigma_p^2}\right],\end{aligned} \tag{2.24}$$

where the effective oscillation and coherence lengths are defined by

$$L_{ab}^{\rm osc}(L) \equiv \frac{4\pi E_0 L}{\int_0^L dx\,\Delta\mu_{ab}^2(x)}\,, \qquad L_{ab}^{\rm coh}(L) \equiv \frac{2\sqrt{2}\sigma_x L}{\left|\int_0^L dx\,\Delta v_{ab}(x)\right|}. \tag{2.25}$$

The problem has reduced to computing the integrals over the effective mass and group velocity differences along the neutrino path.

Let us comment on the following issues in connection with eqs. (2.24) and (2.25):

1. The fact that the effective mixing matrices depend on the production and detection location is one part of the well-known MSW effect [21, 31].

2. Compared to the usual oscillation and coherence lengths, $L^{\rm osc} = 4\pi E_0/\Delta_0, L^{\rm coh} = 2\sqrt{2}\sigma_x/|\Delta v_0|$ (where $\Delta_0 (\Delta v_0)$ is the mass squared (velocity) difference in vacuum), we see that the corresponding effective lengths take into account the changes of $\Delta\mu_{ab}^2(x)$ and $\Delta v_{ab}(x)$ over the whole path of propagation. In this sense the oscillation and coherence lengths are not local quantities anymore.

3. The physical coherence length, corresponding to a length scale where the oscillation ceases, can be obtained by solving the equation $L = L_{ab}^{\rm coh}(L)$. The physical oscillation length for variable density lacks a clear definition.

4. The last term of the exponential in eq. (2.24) is related to the energy conservation within the uncertainty σ_p. Its physical meaning is easy to understand: if e.g. $|E_1 - E_2| \gg \sigma_p$, only one of the states ν_1 or ν_2 is "allowed", i.e. there is no oscillation.

5. The treatment presented here is not limited to any specific density profile if only eq. (2.5) is valid (i.e. the adiabaticity is in effect, the matter density is not too high etc.).

The discussion of ref. [14] is in many respects applicable also here.

We finally point out that the calculation of $\int_0^L dx\, \Delta v_{ab}(x)$ is trivial in the relativistic limit: the group velocity is given by definition by

$$v_a(x) = \frac{\partial E_a}{\partial p_a} \approx 1 - \frac{\mu_a^2(x)}{2\, p_a^2(x)} + \frac{1}{2\, p_a(x)} \frac{\partial \mu_a^2}{\partial p_a}\,, \tag{2.26}$$

and taking into account the approximation of eq. (2.22)

$$v_a(x) \approx 1 - \frac{\mu_a^2(x)}{2E_0^2} + \frac{\partial_{E_0} \mu_a^2}{2E_0}\,. \tag{2.27}$$

Hence

$$\int_0^L dx\, \Delta v_{ab}(x) \approx \frac{1}{2E_0^2} \left(-1 + E_0\, \partial_{E_0}\right) \int_0^L dx\, \Delta\mu_{ab}^2(x)\,, \tag{2.28}$$

i.e. we see that $\int_0^L dx\, \Delta v_{ab}(x)$ can be obtained easily if $\int_0^L dx\, \Delta\mu_{ab}^2(x)$ is known.

3. Examples

As an example we calculate the effective oscillation and coherence lengths for two specific density profiles. For simplicity only two neutrino flavors, ν_e and ν_μ, are considered.

The effective mass squared difference in matter is given by [24]

$$\Delta_m(x) = \sqrt{\Big(\Delta_0 \cos 2\theta - 2\sqrt{2}\, E_0\, G_{\rm F} N_e(x)\Big)^2 + \Delta_0^2 \sin^2 2\theta}\,, \tag{3.1}$$

where Δ_0 is the mass squared difference in vacuum, θ is the vacuum mixing angle, $G_{\rm F}$ is the Fermi constant and $N_e(x)$ is the number density of electrons. (For other neutrino flavors or exotic matter contents $N_e(x)$ should be replaced by an appropriate combination of the particle densities of the matter.) The resonance, where the effective mixing is maximal, is reached at the density

$$N_e(x_R) = \frac{\Delta_0 \cos 2\theta}{2\sqrt{2}\, E_0\, G_{\rm F}}\,. \tag{3.2}$$

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The passage through the resonance is adiabatic when

$$Q \equiv \frac{\Delta_0 \sin^2 2\theta}{E_0 \cos 2\theta} \left| \frac{N_e(x_R)}{N_e'(x_R)} \right| \gg 1\,, \tag{3.3}$$

where Q is the adiabaticity parameter. It is also assumed that $|G_{\rm F} N_e(x)| \ll E_0$.

One should notice that the matter may affect considerably the oscillation of neutrinos propagating in constant density. This is the case e.g. if N_e is very close to its resonance value and θ is small. Then (see eq. (3.1)) $\Delta_m \ll \Delta_0$ and the oscillation length may be highly longer than in the vacuum case. Since at the surface of earth $\rho \sim 3\,{\rm g/cm}^3$ and $Y_e \approx 1/2$, and hence $G_{\rm F} N_e \sim 10^{-13}\,{\rm eV}$, the effect might be observable in long baseline experiments e.g. for $E_0 \sim 10\,{\rm GeV}$, $\Delta_0 \sim 10^{-3}\,{\rm eV}^2$ (and small θ). Also the group velocity difference, and consequently the coherence length, can be modified for suitable parameter values (cf. [10, eqs. (15) and (16)]). At the resonance the coherence length is usually increased.

3.1 Linear density profile

The linear profile is by far the most important example, because many actual profiles can be locally approximated by it. Let us parameterize the density as

$$N_e(x) = \lambda(\kappa - x)\,, \tag{3.4}$$

where λ and κ are parameters. The adiabaticity condition, eq. (3.3), leads to

$$\frac{(\Delta_0 \sin 2\theta)^2}{E_0^2\, G_{\rm F} \lambda} \gg 1\,. \tag{3.5}$$

We first calculate

$$\int_0^L dx\, \Delta_m(x) = \frac{1}{4c} \Big[I(\kappa) - I(\kappa - L) \Big]\,, \tag{3.6}$$

with (see e.g. [32])

$$\begin{aligned} I(x) = {} & (2\,c\,x + b)\sqrt{a + b\,x + c\,x^2} + \\ & + \frac{4\,a\,c - b^2}{2\sqrt{c}} \ln \Big(2\sqrt{c\,(a + b\,x + c\,x^2)} + 2\,c\,x + b \Big)\,, \end{aligned} \tag{3.7}$$

and $a = \Delta_0^2$, $b = -4\sqrt{2}\, E_0\, G_{\rm F} \lambda\, \Delta_0 \cos 2\theta$, $c = 8\,(E_0\, G_{\rm F} \lambda)^2$. A direct substitution to eq. (2.25) then yields the effective oscillation length, in principle. Using eqs. (2.25), (2.28) and (3.6), the effective coherence length could also be calculated, but here that tedious calculation is omitted.

Since eq. (3.6) is not very illustrative, we will next consider the low density limit (i.e. small λ). One has

$$\int_0^L dx \Delta_m(x) = L\,\Delta_0 + \sqrt{2}\, E_0\, G_{\rm F} \lambda \cos 2\theta \left(L^2 - 2\kappa L \right) + O(\lambda^2) \tag{3.8}$$

yielding

$$L^{\mathrm{osc}} \approx \frac{4\pi E_0}{\Delta_0}\left[1-\frac{\sqrt{2}\,E_0\,G_{\mathrm{F}}\lambda\cos 2\theta}{\Delta_0}\,(L-2\kappa)\right], \tag{3.9}$$

where the second term in the brackets can be regarded as the first-order correction due to matter. The coherence length is to this order (with eqs. (2.25) and (2.28))

$$L^{\mathrm{coh}} \approx \frac{4\sqrt{2}\,\sigma_x\,E_0^2}{|\Delta_0|}, \tag{3.10}$$

i.e. the same as in vacuum. The lowest-order correction of the coherence length is in fact always of the second order: the linear term $\sim E_0\,N_e(x)$ (see eq. (3.1)) is wiped out by the "operator" $-1+E_0\,\partial_{E_0}$ in eq. (2.28).

3.2 Exponential density profile

Let the density of electrons be given by

$$N_e(x) = \lambda\, e^{-\kappa x}, \tag{3.11}$$

where λ and κ are parameters. eq. (3.3) yields in this case

$$\frac{\Delta_0 \sin^2 2\theta}{E_0 \cos 2\theta\kappa} \gg 1\,. \tag{3.12}$$

Now we can write [32]

$$\int_0^L dx\,\Delta_m(x) = \frac{1}{\kappa}\left[I(1)-I\left(e^{-\kappa L}\right)\right], \tag{3.13}$$

where

$$\begin{aligned} I(x) =& \sqrt{a+bx+cx^2}-\sqrt{a}\ln\left(\frac{2a+b\,x+2\sqrt{a(a+b\,x+c\,x^2)}}{x}\right) \\ &+\frac{b}{2\sqrt{c}}\ln\left(2\sqrt{c\,(a+b\,x+c\,x^2)}+2c\,x+b\right), \end{aligned} \tag{3.14}$$

and $a=\Delta_0^2,\ b=-4\sqrt{2}\,E_0\,G_{\mathrm{F}}\lambda\,\Delta_0\cos 2\theta,\ c=8(E_0\,G_{\mathrm{F}}\lambda)^2$. Once again, this result with eqs. (2.25) and (2.28) would allow us to obtain the effective oscillation and coherence lengths in principle.

In the low density limit we can expand

$$\begin{aligned} \int_0^L dx\,\Delta_m(x) =& L\,\Delta_0+\frac{2\sqrt{2}E_0\,G_{\mathrm{F}}\lambda\cos 2\theta}{\kappa}\left(e^{-\kappa L}-1\right)- \\ &-\frac{2(E_0\,G_{\mathrm{F}}\lambda\sin 2\theta)^2}{\Delta_0\kappa}\left(e^{-2\kappa L}-1\right)+O\left(\lambda^3\right). \end{aligned} \tag{3.15}$$

Now we get the first-order correction to the effective oscillation length

$$L^{\rm osc} \approx \frac{4\pi E_0}{\Delta_0}\left[1 - \frac{2\sqrt{2}\,E_0\,G_{\rm F}\lambda\cos 2\theta}{L\Delta_0\kappa}\left(e^{-\kappa L}-1\right)\right]. \tag{3.16}$$

Equations (2.25) and (2.28) yield the correction to the effective coherence length

$$L^{\rm coh} \approx \frac{4\sqrt{2}\,\sigma_x E_0^2}{|\Delta_0|}\left[1 - \frac{2(E_0\,G_{\rm F}\lambda\sin 2\theta)^2}{L\,\Delta_0^2\kappa}\left(e^{-2\kappa L}-1\right)\right], \tag{3.17}$$

which is of the second order, in accordance with the remark made in the previous example.

We conclude with a simple numerical application on solar neutrinos. The electron number density in the sun is approximately [33]

$$N_e(x) = 245\,{\rm N_A}\exp\left(-10.54\,\frac{x}{R_\odot}\right)\,{\rm cm}^{-3}\,, \tag{3.18}$$

where $\rm N_A$ is Avogadro's number. Hence (eq. (3.11)) $\lambda \sim 10^{12}\,{\rm eV}^3$ and $\kappa\,L \sim 10^3$ if neutrinos are produced in the center of the sun and detected on earth. We use the values $\Delta_0 \sim 10^{-4}\,{\rm eV}^2$ and $E_0 \sim 1\,{\rm MeV}$, which fulfill the adiabaticity condition, eq. (3.12), even for rather small values of θ. eq. (3.16) yields the correction of the effective oscillation length

$$\frac{E_0\,G_{\rm F}\lambda\cos 2\theta}{L\,\Delta_0\kappa} \sim 10^{-4}\cos 2\theta\,, \tag{3.19}$$

where $G_{\rm F} \sim 10^{-23}\,{\rm eV}^{-2}$. The correction of the effective coherence length is similarly

$$\frac{(E_0\,G_{\rm F}\lambda\sin 2\theta)^2}{L\,\Delta_0^2\kappa} \sim 10^{-5}\sin^2 2\theta\,, \tag{3.20}$$

where eq. (3.17) was used. It is thus seen that the matter effect modifies the oscillation and coherence lengths of solar neutrinos only slightly at least within the framework of this example. This fact is, of course, due to the insignificance of the density outside the sun, i.e. the neutrinos propagate mainly in vacuum between the sun and earth. Similarly, it is to be believed that the matter effect is unimportant for the oscillation of all extraterrestrial neutrinos (excluding, of course, MSW effect and parametric resonance).

We finally call your attention to the fact that even though both the linear and exponential density profiles violate the condition $|G_{\rm F}N_e(x)| \ll E_0$ for some values of x, the sharpness of the wave packets forces x to be situated in harmless region (see eqs. (2.12) and (2.13)). It is due to this same reason that the profile of eq. (3.18), written in spherical coordinates, is directly applicable. A more obvious reason is seen in the integrals in eq. (2.25).

4. Non-adiabatic neutrino propagation in matter

4.1 Solution of the equation of motion for two neutrino flavors

In the previous sections we have assumed that the neutrino propagation is adiabatic. In most physical environments, however, the matter density may change so rapidly that the nonadiabaticity must be taken into account. When neutrinos propagate nonadiabatically the local matter "eigenstates" are not anymore eigenstates but they become mixed. This means that transitions (i.e. level crossings) between the matter states may occur at certain probability. Mathematically this fact manifests itself in the complete equation of motion (in the matter basis), which includes also non-diagonal terms, whereas in the adiabatic limit the corresponding equation is diagonal (cf. eq. (2.5)).

In this section we look for a formal solution to the two flavor equation of motion [24, 28, 29, 34, 35]

$$i\,\partial_x \begin{pmatrix} \psi_1(x,E) \\ \psi_2(x,E) \end{pmatrix} = \begin{pmatrix} -\frac{\Delta(x)}{4E} & -i\,\theta'_m(x) \\ i\,\theta'_m(x) & \frac{\Delta(x)}{4E} \end{pmatrix} \begin{pmatrix} \psi_1(x,E) \\ \psi_2(x,E) \end{pmatrix}, \tag{4.1}$$

where $\Delta(x) \equiv \mu_2^2(x) - \mu_1^2(x)$ (this is not necessarily the same as in eq. (3.1) if the intermediate matter contains e.g. muons), $\theta'_m(x) = \partial_x\theta_m(x)$ and $\theta_m(x)$ is the mixing angle in matter. In the limit $4E|\theta'_m(x)| \ll \Delta(x)$, equivalent to the previously considered adiabaticity condition, eq. (3.3) (for $\nu_e \leftrightarrow \nu_\mu$ oscillation), the non-diagonal terms can be neglected, and eqs. (4.1) and (2.5) (with $a = 1, 2$) correspond to each other perfectly (discarding terms proportional to the identity matrix). Level crossings, on the other hand, are obviously caused by the non-diagonal terms $\pm i\theta'_m(x)$. These issues, as well as exact calculations of the level crossing probabilities for specific density profiles, have been extensively discussed by numerous authors; see e.g. [10], [24]–[29] and [34]–[46].

One should remember that eq. (4.1) describes the physical situation accurately enough if the following conditions hold (see e.g. ref. [22]):

1. Neutrinos are relativistic.

2. The density is low enough, i.e. $|G_{\mathrm{F}}N(x)| \ll E$, where $N(x)$ is the number density of particle species relevant for a case under consideration (mentioned already in the previous section for $N(x) = N_e(x)$).

3. The density must not change appreciably over a length scale equal to the neutrino's de Broglie wavelength.

The expression "arbitrary density profile", used in the following, is to be understood in the context of the abovementioned limitations.

We now demonstrate how eq. (4.1) can be solved. Defining (we omit here the energy dependence of the wave functions)

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \tag{4.2}$$

one has

$$i\,\partial_x \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 0 & B(x) \\ B^*(x) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{4.3}$$

where $B(x) \equiv -\frac{\Delta(x)}{4E} - i\,\theta'_m(x)$ (and $B^*(x) = -\frac{\Delta(x)}{4E} + i\,\theta'_m(x)$). The solution of eq. (4.3) is obtainable after some effort:

$$\begin{aligned} \phi_1(x) &= C_1\,\Xi_-(B,B^*) + C_2\,\Xi_+(B,B^*)\,, \\ \phi_2(x) &= C_1\,\Xi_-(B^*,B) - C_2\,\Xi_+(B^*,B)\,, \end{aligned} \tag{4.4}$$

where C_1 and C_2 are constants, and

$$\Xi_\pm(Y,Z) \equiv 1 \pm i \int Y - \int Y \int Z \mp i \int Y \int Z \int Y + \int Y \int Z \int Y \int Z \pm \cdots \tag{4.5}$$

with e.g. $\int Y \int Z \int Y \equiv \int_{x_0}^{x} dx_1\, Y(x_1) \int_{x_0}^{x_1} dx_2\, Z(x_2) \int_{x_0}^{x_2} dx_3\, Y(x_3)$. Equation (4.2) yields finally

$$\begin{aligned} \psi_1(x,E) &= \frac{1}{2}\,\psi_1(x_0,E)\Big(\Xi_-(B,B^*) + \Xi_-(B^*,B)\Big) - \\ &\quad -\frac{1}{2}\,\psi_2(x_0,E)\Big(\Xi_+(B,B^*) - \Xi_+(B^*,B)\Big)\,, \\ \psi_2(x,E) &= \frac{1}{2}\,\psi_1(x_0,E)\Big(- \Xi_-(B,B^*) + \Xi_-(B^*,B)\Big) + \\ &\quad +\frac{1}{2}\,\psi_2(x_0,E)\Big(\Xi_+(B,B^*) + \Xi_+(B^*,B)\Big)\,. \end{aligned} \tag{4.6}$$

A straightforward substitution shows that this is really the solution of eq. (4.1). As far as we know, no such complete solution has previously been presented in the literature. It is to be emphasized that eq. (4.6) applies to arbitrary density profile, but on the other hand it has the undeniable deficiency of being somewhat formal and perhaps not very useful for practical calculations. Further discussion on the solution of eq. (4.1) and on the corresponding equation in the flavor basis is found in appendix B.

Let us show briefly that our solution gives a meaningful result in two specific cases. In *the adiabatic limit* $B(x) \approx -\frac{\Delta(x)}{4E} \approx B^*(x)$ and hence

$$\psi_1(x,E) = \psi_1(x_0,E)\,\Xi_-(B,B) = \psi_1(x_0,E)\exp\left(i\int_{x_0}^{x} dx'\,\frac{\Delta(x)}{4E}\right), \tag{4.7}$$

and similarly

$$\psi_2(x,E) = \psi_2(x_0,E)\exp\left(-i\int_{x_0}^{x} dx'\,\frac{\Delta(x)}{4E}\right), \tag{4.8}$$

i.e. the correct result is obtained. In *the extremely non-adiabatic limit*[1], on the other hand, $B(x) \approx -i\,\theta'_m(x) \approx -B^*(x)$ (in the resonance region), yielding

$$\begin{aligned}\psi_1(x,E) &= \psi_1(x_0,E)\left(1+\int B\int B+\cdots\right)- \\ &\quad -\psi_2(x_0,E)\left(i\int B+i\int B\int B\int B+\cdots\right), \\ \psi_2(x,E) &= \psi_1(x_0,E)\left(i\int B+i\int B\int B\int B+\cdots\right)+ \\ &\quad +\psi_2(x_0,E)\left(1+\int B\int B+\cdots\right). \end{aligned} \tag{4.9}$$

One may assume that $\theta_m(x)$ changes abruptly from $\pi/2$ to θ (the vacuum mixing angle) in the resonance, i.e.

$$\int_{x_0}^{x} B \approx -i\left(\theta-\frac{\pi}{2}\right) \tag{4.10}$$

if $x_0(x)$ is before (after) the resonance. Hence

$$\begin{aligned}\psi_1(x,E) &= \psi_1(x_0,E)\cos\left(\theta-\frac{\pi}{2}\right)-\psi_2(x_0,E)\sin\left(\theta-\frac{\pi}{2}\right), \\ \psi_2(x,E) &= \psi_1(x_0,E)\sin\left(\theta-\frac{\pi}{2}\right)+\psi_2(x_0,E)\cos\left(\theta-\frac{\pi}{2}\right). \end{aligned} \tag{4.11}$$

Putting e.g. $\psi_1(x_0,E)=0,\ \psi_2(x_0,E)=1$, we see that the level crossing probability is

$$|\psi_1(x,E)|^2 = \sin^2\left(\theta-\frac{\pi}{2}\right) = \cos^2\theta\,, \tag{4.12}$$

as it should in this limit [29].

4.2 Level crossing probabilities and wave packets

We now present how the existing results on the level crossing probabilities can be combined with the wave packet description in a consistent manner. As in the previous section, we restrict to two flavors which is sufficient for understanding the relevant phenomena.

Consider a neutrino that propagates initially as ψ_2 and has the usual gaussian form (eq. (2.8)), i.e.

$$\psi_1(x_0,E)=0 \qquad \text{and} \qquad \psi_2(x_0,E)=N\exp\left[-\frac{(E-E_2)^2}{4\sigma_{EP}^2}\right], \tag{4.13}$$

[1]Here we consider the usual textbook example where the matter states are related to electron and muon flavor states, and the matter does not contain muons. Remember that eq. (4.1) is not necessarily limited to this standard case.

where $N = (2\pi\sigma_{EP}^2)^{-1/4}$. From eq. (4.6) it follows that

$$\begin{aligned}\psi_1(x,E) &= -\frac{N}{2}\exp\left[-\frac{(E-E_2)^2}{4\sigma_{EP}^2}\right] f\,(x_0,x,E)\,,\\ \psi_2(x,E) &= \frac{N}{2}\exp\left[-\frac{(E-E_2)^2}{4\sigma_{EP}^2}\right] h\,(x_0,x,E)\,,\end{aligned}\tag{4.14}$$

where $f(x_0,x,E)$ and $h(x_0,x,E)$ represent the corresponding expansions in eq. (4.6). Hence

$$\psi_1(x,t) = -\frac{N}{2\sqrt{2\pi}}\int dE\exp\left[-i\,E\,t-\frac{(E-E_2)^2}{4\sigma_{EP}^2}\right] f\,(x_0,x,E)\,,\tag{4.15}$$

where eq. (2.2) was used. Defining

$$\exp\left[-\frac{(E-E_2)^2}{4\sigma_{EP}^2}\right] f\,(x_0,x,E) \equiv g(x_0,x,E)\tag{4.16}$$

one has

$$\psi_1(x,t) = -\frac{N}{2}\,\hat{g}\,(x_0,x,t)\,,\tag{4.17}$$

where the circumflex stands for the Fourier transform. The level crossing probability is $|\psi_1(x,t)|^2$ but we integrate over time (cf. section 2) and have

$$\begin{aligned}|\psi_1(x)|^2 &= \int dt\,|\psi_1(x,t)|^2 = \frac{N^2}{4}\int dt\,|\hat{g}(x_0,x,t)|^2\\ &= \frac{N^2}{4}\int dE\,|g(x_0,x,E)|^2\,,\end{aligned}\tag{4.18}$$

where Parseval's identity was used. We can express the level crossing probability for wave packets using the respective probability for a plane wave, $P_{\rm lc}(E) = \frac{1}{4}\,|f\,(x_0,x,E)|^2$. Hence

$$P_{\rm lc}\,(E_2,\sigma_{EP}) \equiv |\psi_1(x)|^2 = N^2\int dE\exp\left[-\frac{(E-E_2)^2}{2\sigma_{EP}^2}\right] P_{\rm lc}(E)\,,\tag{4.19}$$

where $P_{\rm lc}\,(E_2,\sigma_{EP})$ is the generalized level crossing probability that takes into account the energy width of the wave packet.

The wave packet effects can be seen more clearly by expanding $P_{\rm lc}\,(E)$ in series. Assuming σ_{EP} to be small (section 2), it is sufficient to take the lowest terms, and eq. (4.19) gives (with $E_2 \to E$)

$$P_{\rm lc}\,(E,\sigma_{EP}) = P_{\rm lc}\,(E) + \frac{\sigma_{EP}^2}{2}\,\frac{\partial^2 P_{\rm lc}(E)}{\partial E^2} + O\left(\sigma_{EP}^4\,\frac{\partial^4 P_{\rm lc}(E)}{\partial E^4}\right).\tag{4.20}$$

This equation clearly indicates that the use of the wave packets modifies the usual level crossing probabilities, $P_{\rm lc}(E)$. It also shows that the wave packet correction is easily calculable if $P_{\rm lc}(E)$ is known.

Let us consider two simple examples:

1. *Linear density profile.* The well-known Landau-Zener probability is
$$P_{\rm LZ}\,(E) = \exp\left(-\frac{\pi}{4}\,Q\right), \tag{4.21}$$
where Q, given in eq. (3.3), should not be too small. Using eq. (4.20) and remembering that $Q \sim 1/E^2$ (see eq. (3.5)) one has
$$P_{\rm LZ}(E,\sigma_{EP}) = \left[1 + \left(\frac{\sigma_{EP}}{2E}\right)^2 \left(\frac{\pi^2}{2}\,Q^2 - 3\,\pi\,Q\right)\right] \exp\left(-\frac{\pi}{4}\,Q\right). \tag{4.22}$$
For the most relevant cases (i.e. $P_{\rm LZ}\,(E)$ not too small) $(\pi Q)^2/2 - 3\,\pi\,Q$ is of the order of unity.

2. *Exponential density profile* ($N_e(x) \sim e^{-\kappa x}$). Now [27, 28, 44]
$$P_{\rm lc}\,(E) = \frac{e^{-\pi\,\delta\,(1-\cos 2\theta)} - e^{-2\pi\delta}}{1 - e^{-2\pi\delta}} = \frac{\sinh(B-A)}{\sinh B}\,e^{-A}\,, \tag{4.23}$$
where
$$\delta = \frac{\Delta_0}{2E\kappa}\,, \qquad A = \frac{\pi\delta}{2}\,(1-\cos 2\theta)\,, \qquad B = \pi\delta\,. \tag{4.24}$$
A tedious calculation gives
$$P_{\rm lc}(E,\sigma_{EP}) = P_{\rm lc}(E)\left[1 + \left(\frac{\sigma_{EP}}{E}\right)^2 \Gamma\right], \tag{4.25}$$
where
$$\begin{aligned}\Gamma = A\Big(1+\coth(B-A)\Big)\Big(A - 1 + B(\coth B - 1)\Big) + \\ + B\Big(\coth(B-A) - \coth B\Big)(1 - B\coth B)\,.\end{aligned} \tag{4.26}$$
Again, for the interesting range of parameters $\Gamma \sim O(1)$ at most when e.g. solar neutrinos are considered (with $E = 1\,\mathrm{MeV}$, $\Delta_0 = 10^{-4}\,\mathrm{eV}^2$, and the density profile as given in eq. (3.18)).

The wave packet corrections turn out to be negligible for small σ_{EP}/E. This happens to be true in most physical environments, e.g. for solar neutrinos $\sigma_{EP}/E \sim 10^{-4} - 10^{-5}$ [12, 24]. If $\sigma_{EP} \sim E$, on the other hand, the simple wave packet treatment is inaccurate [24]. This fact is manifest also in our calculation: if σ_{EP} is not small enough, the integration in eq. (4.19) becomes problematic since $P_{\rm lc}(E)$ is not necessarily meaningfully defined for negative E values. Anyway, there might be at least in principle a situation where the energy distribution of the neutrino wave function deviates considerably from a plane wave. Our calculation suggests that then the usual level crossing probabilities would not be totally reliable.

Finally, it is to be noted that the use of some specific $P_{\rm lc}(E)$ restricts the location of x_0 and x in eqs. (4.6) and (4.14); $P_{\rm LZ}(E)$, for example, is valid only if $x_0(x)$ is situated well before (after) the resonance. If, on the contrary, x_0 and/or x are/is in the resonance region, $P_{\rm LZ}(E)$ cannot be used.

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5. Summary

We applied the wave packet formalism in order to study neutrino oscillations in matter in the adiabatic limit, and found out that the effective oscillation and coherence lengths take into account the whole path the neutrino has traversed. Results for the linear and exponential density profiles were briefly presented. The corrections for the predictions of observable fluxes of solar neutrinos seem to be quite small. On the other hand, the matter may affect significantly the oscillation of the neutrinos e.g. in long baseline experiments for suitable values of parameters.

We then considered the equation of motion for two neutrino flavors, and managed to solve it formally for arbitrary density profile. Our method clearly applies to any differential equation of the same form.

Finally, we showed that the level crossing probabilities of the wave packets differ from those of plane waves. The difference is practically equivalent to a simple average over energy (cf. eq. (4.19)), not essentially related to the separation of wave packets. The finite width of the wave packet does not result in any observable effect for the physical situations we have considered.

We could have continued our work by combining the wave packet treatment of section 2 with the complete solution of the two flavor equation in section 4.1. In that case the solution of eq. (2.5) (for $a = 1, 2$) should be replaced by eq. (4.6) in order to correctly take into account the effects due to nonadiabaticity. However, the presented results suggest that this would not reveal any new physics, so the calculation has been omitted.

In this work we used a model that in principle is more physical and hence more accurate than the models used normally. On the other hand, since many of the calculations in this framework are quite complicated, we considered the limits of validity of the simpler plane wave approaches. Our results show that the present neutrino observations and the phenomena behind them can be described by a plane wave model accurately enough, regarding the current precision of the experiments.

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A. Integration over x in eq. (2.14)

We can write eq. (2.14) as

$$A_{\alpha\beta}\,(L,T) \sim C(L,T) \int dx\, e^{F(x)}\,, \tag{A.1}$$

where

$$F(x) \equiv -\frac{v_a^2(0)}{4\sigma_{xP}^2}\left(\int_0^x \frac{dx'}{v_a(x')} - T\right)^2 - \frac{v_a^2(L)}{4\,\sigma_{xD}^2}\left(\int_L^x \frac{dx'}{v_a(x')}\right)^2 . \tag{A.2}$$

The definition of the saddle point is in turn $F'(x_0) = 0$ ($F'(x) = \frac{d}{dx}F(x)$), leading to

$$\sigma_{xD}^2\left(\int_0^{x_0} \frac{dx'}{v_a(x')} - T\right) v_a^2(0) + \sigma_{xP}^2 \int_L^{x_0} \frac{dx'}{v_a(x')}\, v_a^2(L) = 0\,. \tag{A.3}$$

From this one can solve that

$$\int_0^{x_0} \frac{dx'}{v_a(x')} = \frac{\sigma_{xD}^2\, v_a^2(0)\, T + \sigma_{xP}^2\, v_a^2(L) \int_0^L \frac{dx'}{v_a(x')}}{\sigma_{xD}^2\, v_a^2(0) + \sigma_{xP}^2\, v_a^2(L)} \tag{A.4}$$

and

$$\int_L^{x_0} \frac{dx'}{v_a(x')} = \frac{\sigma_{xD}^2\, v_a^2(0)\left(T - \int_0^L \frac{dx'}{v_a(x')}\right)}{\sigma_{xD}^2\, v_a^2(0) + \sigma_{xP}^2\, v_a^2(L)}\,, \tag{A.5}$$

and finally after some algebra

$$F(x_0) = -\frac{1}{4}\,\frac{\left(T - \int_0^L \frac{dx}{v_a(x)}\right)^2}{\left(\frac{\sigma_{xP}}{v_a(0)}\right)^2 + \left(\frac{\sigma_{xD}}{v_a(L)}\right)^2}\,. \tag{A.6}$$

Let us now examine the higher derivatives of $F(x)$. The second derivative is

$$\begin{aligned} F''(x) = &-\frac{1}{2\sigma_{xP}^2}\left(\frac{v_a(0)}{v_a(x)}\right)^2 + \frac{1}{2\sigma_{xP}^2}\left(\int_0^x \frac{dx'}{v_a(x')} - T\right)\left(\frac{v_a(0)}{v_a(x)}\right)^2 v_a'(x) - \\ &-\frac{1}{2\sigma_{xD}^2}\left(\frac{v_a(L)}{v_a(x)}\right)^2 + \frac{1}{2\sigma_{xD}^2}\left(\int_L^x \frac{dx'}{v_a(x')}\right)\left(\frac{v_a(L)}{v_a(x)}\right)^2 v_a'(x)\,. \end{aligned} \tag{A.7}$$

In the relativistic limit $v_a(x) = 1 + O(\mu_a^2/E^2)$, and eq. (A.7) reduces to[2]

$$F''(x_0) = -\frac{1}{2\sigma_{xP}^2} - \frac{1}{2\sigma_{xD}^2} + O\left(\frac{L-T}{\sigma_{xP,D}^2}\,\frac{d}{dx_0}\,\frac{\mu_a^2(x_0)}{E^2}\right), \tag{A.8}$$

where effectively $\sigma_{xP,D}^2 = \max\{\sigma_{xP}^2, \sigma_{xD}^2\}$. From eq. (A.7) one sees also that the third derivative of $F(x)$ includes terms proportional to $v_a'(x)$, $(v_a'(x))^2$ and $v_a''(x)$. It is thus obvious that $F'''(x)$ and all the higher derivatives are negligible in the relativistic limit, and we can approximate (with $F'(x_0) = 0$)

$$F(x) \approx F(x_0) + \frac{1}{2}\,F''(x_0)(x - x_0)^2\,. \tag{A.9}$$

The relevant part of the integral of eq. (2.14) becomes

$$\int dx\, e^{F(x)} \approx e^{F(x_0)} \int dx \exp\left[-\frac{1}{4}\left(\frac{1}{\sigma_{xP}^2} + \frac{1}{\sigma_{xD}^2}\right)(x - x_0)^2\right], \tag{A.10}$$

where the final gaussian integral gives only an unimportant numerical factor.

[2]The adiabaticity further reinforces the smallness of $v_a'(x)$.

B. Additional calculations in connection with the two flavor equation of motion

We present another way of solving eq. (4.1). Instead of eq. (4.2), we now define (compare to ref. [43])

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} \exp\left(i\int_{x_0}^x dx' \frac{\Delta(x')}{4E}\right) & 0 \\ 0 & \exp\left(-i\int_{x_0}^x dx' \frac{\Delta(x')}{4E}\right) \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{B.1}$$

leading to

$$i\,\partial_x \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 0 & -i\,D(x) \\ i\,D^*(x) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{B.2}$$

where

$$D(x) \equiv \theta'_m(x) \exp\left(-i\int_{x_0}^x dx' \, \frac{\Delta(x')}{2E}\right). \tag{B.3}$$

Equation (B.2) yields easily

$$\begin{aligned}
\phi_1(x) = C_1 \Big(1 - \int D - \int D \int D^* + \int D \int D^* \int D + \\
+ \int D \int D^* \int D \int D^* + \cdots\Big) + \\
+ C_2 \Big(1 + \int D - \int D \int D^* - \int D \int D^* \int D + \\
+ \int D \int D^* \int D \int D^* + \cdots\Big), \\
\phi_2(x) = C_1 \Big(1 + \int D^* - \int D^* \int D - \int D^* \int D \int D^* + \\
+ \int D^* \int D \int D^* \int D + \cdots\Big) - \\
- C_2 \Big(1 - \int D^* - \int D^* \int D + \int D^* \int D \int D^* + \\
+ \int D^* \int D \int D^* \int D + \cdots\Big),
\end{aligned} \tag{B.4}$$

where the notation is as before (see eq. (4.5)). Finally one has

$$\begin{aligned}
\psi_1(x) = \exp\left(i\int_{x_0}^x dx' \frac{\Delta(x')}{4E}\right) \times \\
\times \Big[\psi_1(x_0)\Big(1 - \int D \int D^* + \int D \int D^* \int D \int D^* + \cdots\Big) + \\
+ \psi_2(x_0)\Big(-\int D + \int D \int D^* \int D + \cdots\Big)\Big],
\end{aligned}$$

$$\psi_2(x) = \exp\left(-i\int_{x_0}^{x} dx'\,\frac{\Delta(x')}{4E}\right)\times$$
$$\times\left[\psi_1(x_0)\left(\int D^* - \int D^*\int D\int D^* + \cdots\right) + \right.$$
$$\left. + \psi_2(x_0)\left(1 - \int D^*\int D + \int D^*\int D\int D^*\int D + \cdots\right)\right], \quad \text{(B.5)}$$

where the energy dependence is not written down explicitly. We thus see that eq. (4.6) is not the only form in which the solution of eq. (4.1) can be expressed.

The equation of motion in the flavor basis is [24, 28] (and many others)

$$i\,\partial_x \begin{pmatrix}\psi_e(x)\\ \psi_\mu(x)\end{pmatrix} = \frac{1}{4E}\begin{pmatrix}-\Delta_0\cos 2\theta + A_c(x) & \Delta_0\sin 2\theta\\ \Delta_0\sin 2\theta & \Delta_0\cos 2\theta - A_c(x)\end{pmatrix}\begin{pmatrix}\psi_e(x)\\ \psi_\mu(x)\end{pmatrix}, \quad \text{(B.6)}$$

where $A_c(x) = 2\sqrt{2}\,E\,G_{\mathrm{F}}\,N_e(x)$ and other notations are obvious. Since the calculation proceeds as above, we omit the details and content ourselves with giving the final answer:

$$\psi_e(x) = \exp\left[\frac{i}{4E}\int_{x_0}^{x} dx'(\Delta_0\cos 2\theta - A_c(x'))\right]\times$$
$$\times\left[\psi_e(x_0)\left(1 - \int F\int F^* + \int F\int F^*\int F\int F^* + \cdots\right) + \right.$$
$$\left. + \psi_\mu(x_0)\left(-i\int F + i\int F\int F^*\int F + \cdots\right)\right],$$
$$\psi_\mu(x) = \exp\left[\frac{-i}{4E}\int_{x_0}^{x} dx'\Big(\Delta_0\cos 2\theta - A_c(x')\Big)\right]\times$$
$$\times\left[\psi_e(x_0)\left(-i\int F^* + i\int F^*\int F\int F^* + \cdots\right) + \right.$$
$$\left. + \psi_\mu(x_0)\left(1 - \int F^*\int F + \int F^*\int F\int F^*\int F + \cdots\right)\right], \quad \text{(B.7)}$$

where

$$F(x) \equiv \frac{\Delta_0\sin 2\theta}{4E}\exp\left[\frac{-i}{2E}\int_{x_0}^{x} dx'\Big(\Delta_0\cos 2\theta - A_c(x')\Big)\right]. \quad \text{(B.8)}$$

References

[1] Super-Kamiokande collaboration, *Evidence for oscillation of atmospheric neutrinos*, *Phys. Rev. Lett.* **81** (1998) 1562 [hep-ex/9807003].

[2] B. Pontecorvo, *Mesonium and antimesonium*, *Zh. Eksp. Theor. Fiz.* **33** (1957) 549, *Sov. Phys. JETP* **6** (1958) 429.

[3] Z. Maki, M. Nakagawa and S. Sakata, *Remarks on the unified model of elementary particles*, *Prog. Theor. Phys.* **28** (1962) 870.

[4] B. Pontecorvo, *Neutrino experiments and the problem of conservation of leptonic charge*, *Zh. Eksp. Theor. Fiz.* **53** (1969) 1717 *Sov. Phys. JETP* **26** (1968) 984.

[5] S.M. Bilenkii and B. Pontecorvo, *Lepton mixing and neutrino oscillations*, *Phys. Rep.* **41** (1978) 225.

[6] C. Giunti, C.W. Kim and U.W. Lee, *When do neutrinos really oscillate?: quantum mechanics of neutrino oscillations*, *Phys. Rev.* **D 44** (1991) 3635.

[7] J. Rich, *The quantum mechanics of neutrino oscillations*, *Phys. Rev.* **D 48** (1993) 4318.

[8] S.M. Bilenky and S.T. Petcov, *Massive neutrinos and neutrino oscillations*, *Rev. Mod. Phys.* **59** (1987) 671.

[9] B. Kayser, *On the quantum mechanics of neutrino oscillation*, *Phys. Rev.* **D 24** (1981) 110.

[10] C. Giunti, C.W. Kim and U.W. Lee, *Coherence of neutrino oscillations in vacuum and matter in the wave packet treatment*, *Phys. Lett.* **B 274** (1992) 87.

[11] C. Giunti, C.W. Kim, J.A. Lee and U.W. Lee, *On the treatment of neutrino oscillations without resort to weak eigenstates*, *Phys. Rev.* **D 48** (1993) 4310 [hep-ph/9305276].

[12] K. Kiers, S. Nussinov and N. Weiss, *Coherence effects in neutrino oscillations*, *Phys. Rev.* **D 53** (1996) 537 [hep-ph/9506271].

[13] C. Giunti, C.W. Kim and U.W. Lee, *When do neutrinos cease to oscillate?*, *Phys. Lett.* **B 421** (1998) 237 [hep-ph/9709494].

[14] C. Giunti and C.W. Kim, *Coherence of neutrino oscillations in the wave packet approach*, *Phys. Rev.* **D 58** (1998) 017301 [hep-ph/9711363].

[15] M. Nauenberg, *Correlated wave packet treatment of neutrino and neutral meson oscillations*, *Phys. Lett.* **B 447** (1999) 23 [hep-ph/9812441].

[16] W. Grimus and P. Stockinger, *Real oscillations of virtual neutrinos*, *Phys. Rev.* **D 54** (1996) 3414 [hep-ph/9603430].

[17] W. Grimus, P. Stockinger and S. Mohanty, *The field-theoretical approach to coherence in neutrino oscillations*, *Phys. Rev.* **D 59** (1999) 013011 [hep-ph/9807442].

[18] L. Stodolsky, *The unnecessary wavepacket*, *Phys. Rev.* **D 58** (1998) 036006 [hep-ph/9802387].

[19] A. Widom and Y.N. Srivastava, *Neutrino flavor oscillations using the Dirac equation*, hep-ph/9608476.

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[20] Y.V. Shtanov, *Space-time description of neutrino flavour oscillations*, *Phys. Rev.* **D 57** (1998) 4418 [hep-ph/9706378].

[21] L. Wolfenstein, *Neutrino oscillations in matter*, *Phys. Rev.* **D 17** (1978) 2369.

[22] A. Halprin, *Neutrino oscillations in nonuniform matter*, *Phys. Rev.* **D 34** (1986) 3462.

[23] P.D. Mannheim, *Derivation of the formalism for neutrino matter oscillations from the neutrino relativistic field equations*, *Phys. Rev.* **D 37** (1988) 1935.

[24] C.W. Kim and A. Pevsner, *Neutrinos in physics and astrophysics*, Harwood Academic, Chur 1993.

[25] D. Nötzold, *Exact analytic solutions for Mikheev-Smirnov-Wolfenstein level crossings*, *Phys. Rev.* **D 36** (1987) 1625.

[26] S. Toshev, *Exact analytical solution of the two neutrino evolution equation in matter with exponentially varying density*, *Phys. Lett.* **B 196** (1987) 170.

[27] S.T. Petcov, *Exact analytic description of two neutrino oscillations in matter with exponentially varying density*, *Phys. Lett.* **B 200** (1988) 373.

[28] A.B. Balantekin, *Exact solutions for matter-enhanced neutrino oscillations*, *Phys. Rev.* **D 58** (1998) 013001 [hep-ph/9712304].

[29] C.W. Kim, S. Nussinov and W.K. Sze, *Nonadiabatic resonant conversion of solar neutrinos in three generations*, *Phys. Lett.* **B 184** (1987) 403.

[30] C. Giunti, C.W. Kim and U.W. Lee, *Comments on the weak states of neutrinos*, *Phys. Rev.* **D 45** (1992) 2414.

[31] S.P. Mikheev and A.Y. Smirnov, *Resonance enhancement of oscillations in matter and solar neutrino spectroscopy*, *Sov. J. Nucl. Phys.* **42** (1985) 913.

[32] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, series and products*, Academic Press, Orlando 1980.

[33] J.N. Bahcall and R.K. Ulrich, *Solar models, neutrino experiments and helioseismology*, *Rev. Mod. Phys.* **60** (1988) 297.

[34] T.K. Kuo and J. Pantaleone, *Nonadiabatic neutrino oscillations in matter*, *Phys. Rev.* **D 39** (1989) 1930.

[35] A.B. Balantekin and J.F. Beacom, *Semiclassical treatment of matter-enhanced neutrino oscillations for an arbitrary density profile*, *Phys. Rev.* **D 54** (1996) 6323 [hep-ph/9606353].

[36] L. Landau, *Phys. Z. Sovjetunion* **2** (1932) 46.

[37] C. Zener, *Nonadiabatic crossing of energy levels*, *Proc. R. Soc. London, Ser. A* **137** (1932) 696.

[38] S.P. Rosen and J.M. Gelb, *Mikheev-Smirnov-Wolfenstein enhancement of oscillations as a possible solution to the solar-neutrino problem*, *Phys. Rev.* **D 34** (1986) 969.

[39] V. Barger, R.J. N. Phillips and K. Whisnant, *Level crossings in solar neutrino oscillations*, *Phys. Rev.* **D 34** (1986) 980.

[40] S.J. Parke, *Nonadiabatic level crossing in resonant neutrino oscillations*, *Phys. Rev. Lett.* **57** (1986) 1275.

[41] P. Pizzochero, *Nonadiabatic level crossing in neutrino oscillations for an exponential solar density profile*, *Phys. Rev.* **D 36** (1987) 2293.

[42] A.B. Balantekin, S.H. Fricke and P.J. Hatchell, *Analytical and semiclassical aspects of matter enhanced neutrino oscillations*, *Phys. Rev.* **D 38** (1988) 935.

[43] J.C. D'Olivo, *Magnus approximation for nonadiabatic transitions and neutrino oscillations in matter*, *Phys. Rev.* **D 45** (1992) 924

[44] M. Bruggen, W.C. Haxton and Y.Z. Qian, *Landau-Zener treatments of solar neutrino oscillations*, *Phys. Rev.* **D 51** (1995) 4028.

[45] S.T. Petcov, *Describing analytically the matter-enhanced two-neutrino transitions in a medium*, *Phys. Lett.* **B 406** (1997) 355 [hep-ph/9910335].

[46] A.B. Balantekin, J.F. Beacom and J.M. Fetter, *Matter enhanced neutrino oscillations in the quasi-adiabatic limit*, *Phys. Lett.* **B 427** (1998) 317 [hep-ph/9712390].